Correlated Tunnelling of Two Electrons through a Barrier in Quantum Wires

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Abstract

We study the tunnelling trough a potential barrier of the system of two quantum correlated particles. The system is considered in one dimension. The interaction with the barrier and between particles is approximated by δ-potentials. Assuming that the particles have the same masses, we reduce the problem to the set of independent planar scattering problems corresponding to two-body flux symmetry with respect to the barrier. In order to solve the problems we apply the method of Sommerfeld-Maluzhinets integral transformation, which requires the solution of a set of functional equations. We show that the sub-problem which is antisymmetric with respect to the center of mass allows the solution in frames of the Bethe ansatz. So we give the exact solution of this problem. For the subproblem which is symmetric we describe the class of solutions for the functional equations and give an approach for obtaining the solution.

1 Introduction.

The properties of low-dimensional semiconductor structures have attracted much attention in the past few years both experimentally and theoretically. Being the quantum objects by nature the 1D and 2D nanostructures can be described by an appropriate Schrödinger equation with the suitable boundary conditions. There exit a huge literature on the subject however the most results obtained deal with the solution of 1-body or 2-body Schrödinger equation and with the physical quantities that can be described in terms of these solutions. To our knowledge there are only few results concerning the 3-body systems in nanostructures while there are a lot of physical phenomena which need the solution of 3-body quantum problem. Let us mention only two of these phenomena: 1) Observation of anomalies in the optical spectra of quantum wells have been attributed to the formation of negatively-charged excitons [20] . Such complexes can arise when exciton E captures one electron to

form a negatively-charged exciton $E + e \to E^-$ [21]. In ref. [21] the properties of such 3-body complexes in 1D quantum wires have been studied by means of exact solutions of 1D 3-body Schrödinger equation with the Calogero type interaction [22]. 2) Since the discovery of the vanishing transmissivity of a tunnel barrier in a 1D quantum wire due to the electron-electron interaction [23] , new interest has emerged in the transport properties of 1D electron systems. Indeed, the influence of the electron correlations shows up strikingly in non-linear current-voltage relations which are investigated experimentally in narrow semiconducting wires [24].

From the mathematical point of view the type 2) of physical phenomena can be described by 3-body Schrödinger equation in one dimension if one treats the local electron interaction via δ-like potential and the quantum barrier (well) as the third infinitely heavy particle which interacts with the electrons by δ-function potential as well. Solving the appropriate 3-body 1D scattering problem one can calculate in terms of scattering amplitudes such properties of tunneling barrier in quantum wire as conductance, current-voltage characteristics, etc. This scattering problem can be solved either numerically or analytically i.e. exact. The present paper is devoted to the exact solution of the problem mentioned above by means of the reduction of the original 3-body Schrödinger equation to some Riemann-Hilbert problem. Using the symmetries of the system we split the latter into two subproblems which are either symmetric with respect to the center of mass or antisymmetric. For the antisymmetric case we construct an exact solution in terms of the Bethe anzatz. For the sub-problem which is symmetric we suggest a new approach which allows to describe the functional classes of the solutions and reduce the problem to some generalized Dubrovin's equations.

The paper is organized as follows. In Sec.2 we briefly describe the mathematical model for the correlated tunneling of two electrons through a barrier. In Sec.3 by means of Sommerfeld-Maluzhinetz transformation we construct the matrix functional equations which have to be solved. In Sec.4 we give an exact solution of antisymmetric sub-problem in terms of the Bethe anzatz. Sec.5 is devoted to the study of the Riemann-Hilbert problem for the symmetric subproblem. Sec.6 contains some concluding remarks.

2 The Model

The model we study here has been suggested by K. Lipszyc [7] in 1980. However, regardless the simple way the problem is posed, no closed-form solution is known until now. Here we shall come closer to the exact solution of the complete problem and present the solution for one of two sub-problems it is reduced to. Since the problem was carefully described in [7], we shall allow us to abandon the details and present here just the topics essential for understanding of the following considerations (terms, notations, etc.).

We consider the system of two quantum particles with the same mass in one dimension interacting via delta-function potential with the strength γ. In addition, each particle is interacting in the same way with the fixed delta-function barrier

with the strength Γ. Let us denote the coordinates of the particles with respect to the barrier by r_1 and r_2, their masses by m_1 and m_2, choose the units so that $m_1 = m_2 = \hbar/2 = 1$. The Srödinger equation for the wave function u of the system can be written in the form:

$$[-\Delta - \Gamma\delta(r_1) - \Gamma\delta(r_2) - \gamma\delta(r_1 - r_2)]u(r_1, r_2) = \lambda u(r_1, r_2). \tag{1}$$

The correct mathematical formulation of this equation and the corresponding scattering problem is given in terms of extension theory [18, 17] and was described for similar problems in [15, 9, 14]. Here we shall come directly to the equivalent boundary value problem. To do this, let us introduce the polar coordinate system for $(r_1, r_2) = (r\cos(\theta), r\sin(\theta))$, denote the function u in polar coordinates by $\tilde{u}$ and define

$$[f](r, \phi) := f(r, \phi + 0) - f(r, \phi - 0).$$

In this terms, the boundary value problem we shall study looks as [1]:

$$-\Delta\tilde{u} = \lambda\tilde{u}$$
$$\left[\tfrac{\partial}{\partial_\phi}\tilde{u}\right](r, \phi) = -r\Gamma\tilde{u}(r, \phi)\Big|_{\phi=0,\pi/2,\pi,3\pi/2} \tag{2}$$
$$\left[\tfrac{\partial}{\partial_\phi}\tilde{u}\right](r, \phi) = -2r\gamma\tilde{u}(r, \phi)\Big|_{\phi=\pi/4,5\pi/4}$$

Taking into account the symmetry of the system we split the solution of (2) in two parts, symmetric and anti-symmetric with respect to the particle interchange:

$$u(r_1, r_2) = u_-(r_1, r_2) + u_+(r_1, r_2)$$
$$u_-(r_1, r_2) = -u_-(r_2, r_1) = (u(r_1, r_2) - u(r_2, r_1))/2 \tag{3}$$
$$u_+(r_1, r_2) = +u_+(r_2, r_1) = (u(r_1, r_2) + u(r_2, r_1))/2$$

The solution of (2) for u_- is trivial since the role of the second boundary condition (the interaction between particles) vanishes being replaced by the Dirichlet boundary condition $u(x, x) = \tilde{u}(r, \pi/4) = 0$. The expression for u_- can be found as:

$$u_-(r_1, r_2) = f_1(r_1)f_2(r_2) - f_1(r_2)f_2(r_1) \tag{4}$$

where $f_{1,2}$ are the solutions of the one particle problem with δ-interaction .

We shall concentrate on the solution of the boundary problem (2) for u_+. Due to the symmetry of u_+ the boundary value problem (2) can be expressed simpler in the centre of mass frame: $x = (r_1 + r_2)/\sqrt{2}; y = (r_1 - r_2)/\sqrt{2}$, where x is the $\sqrt{2}$ times coordinate of the centre of mass and y is the distance between particles divided by $\sqrt{2}$. Unless otherwise stated we shall use this coordinate system for u_+, the function $\tilde{u}_+$ will represent the same function in polar coordinates. In this notations, one has:

$$-\Delta\tilde{u}_+ = \lambda\tilde{u}_+$$
$$\left[\tfrac{\partial}{\partial_\phi}\tilde{u}_+\right](r, \phi) = -r\Gamma\tilde{u}_+(r, \phi)\Big|_{\phi=\pi/4,3\pi/4}$$
$$\tfrac{\partial}{\partial_\phi}\tilde{u}_+(r, \phi) = -r\gamma\tilde{u}_+(r, \phi)\Big|_{\phi=0,\pi} \tag{5}$$
$$\phi \in (0, \pi)$$

The problem (5) can be further simplified using its symmetry with respect to the transformation $x \rightarrow -x$ ($\phi \rightarrow \pi - \phi$). Let us split the function u_+ accordingly:

$$u_+(x, y) = v_-(x, y) + v_+(x, y)$$
$$v_-(x, y) = -v_-(-x, y) = (u_+(x, y) - u_+(-x, y))/2 \qquad (6)$$
$$v_+(x, y) = +v_+(-x, y) = (u_+(x, y) + u_+(-x, y))/2.$$

The representation (6) splits (5) into two independent problems for $v_\pm$:

$$\left[\frac{\partial}{\partial \phi} v_\pm\right](r, \pi/4) = -r\Gamma v_\pm(r, \pi/4)$$
$$\frac{\partial}{\partial \phi} v_\pm + (r, 0) = -r\gamma v_\pm(r, 0)$$
$$1) \ v_-(r, \pi/2) = 0; \quad 2) \ \frac{\partial}{\partial \phi} v_+(r, \pi/2) = 0 \qquad (7)$$
$$\phi \in (0, \pi/2)$$

These are the boundary value problems we shall deal with in the rest of this paper. In order to be rigorous we must add that we shall investigate the corresponding scattering problems. That is we shall look for the solutions of (7) being the combination of one incoming plane wave $e^{-ikr\cos(\phi-\phi_0)}$, $\phi_0 \in (0, \pi)$, a set of outgoing plane waves, $\phi_0 \notin (0, \pi)$, and a function satisfying the standard scattering conditions at $r \rightarrow \infty$. The detailed description of how the scattering problem should be posed can be found in extensive literature on the subject [1, 2, 4, 5, 7, 9, 17, 18].

As a conclusion to this section, let us briefly discuss the physical meaning of the symmetries we applied. The best visualization can be achieved if we discuss the meaning of the corresponding incoming plane waves. The plane wave in the system (2) corresponds to the incoming, that is in which the particles are directed towards each other and towards the barrier, flow of pairs (since we consider the interaction in pairs, we should consider the pair of particles as one) of particles in which the momenta of the corresponding particles are the same. That is all particles of type 1 have the momentum k_1 and all particles of type 2 have the momentum k_2. We should mention that though the particles are assumed to have the same masses and interact with the barrier in the same way, they can, in principle, differ by some other properties (say by spin). The plane wave in the system (5) corresponds to two fluxes which have the momenta or particles interchanged and have the amplitudes which are either the same or differ by -1. In the case of absolutely identical particles, this would correspond to their statistics. In the last case, the plane wave in the system (7) corresponds to four fluxes coming from two different sides to the barrier, by two from each side, and whose amplitudes are either the same or differ by -1. From the physical point of view, this situation as separate seems to be hardly experimentally realizable but may also give a better understanding of the process. In this paper we give the exact solution of the scattering problem (7) with boundary condition 1) and approach the solution of the (7) with condition 2).

3 Sommerfeld-Maluzhinetz Integral Transformation and The Functional Equations.

To discuss the solutions for (7), K.Lipszyc [7] used the method of Sommerfeld-Maluzhinetz (SM) Integral Transformation [10]. The Maluzhinets type functional equations were derived, however the solution was said to require some new unknown functional class to be used. Thus the problem was closed until further progress in functional analysis. Here we shall also use the SM transformation, however we shall study the functional equations in the form of McGuire [2, 5]. This way of using the SM transformation was effectively applied to several scattering problems [9, 14, 15]. In our previous papers [13, 1] we proved several general statements which described the functional classes needed to solve some SM functional equations (the McGuire type) and the way one can construct their solutions. Here we apply these statements to the concrete problem in question.

Let us apply the SM transformation to the function $v(r, \phi)$ considered on two domains: $\tilde{v}(r, \phi) = v(r, \phi), \phi \in (0, \pi/4)$ and $\hat{v}(r, \phi) = v(r, \phi + \pi/4), \phi \in (0, \pi/4)$.

$$\begin{aligned}
\tilde{v}(r, \phi) &= \int_c e^{-ikr \cos \alpha} \left(g_1(\alpha + \phi) + g_2(\alpha + \pi/4 - \phi) \right) d\alpha \\
\hat{v}(r, \phi) &= \int_c e^{-ikr \cos \alpha} \left(g_3(\alpha + \phi) + g_4(\alpha + \pi/4 - \phi) \right) d\alpha
\end{aligned} \tag{8}$$

where c stands for the standard Sommerfeld contour [10, 9, 13]. Using the representation (8) and the Maluzhinets theorem [10] the boundary value problem (7) can be transformed into a set of functional equations:

$$\begin{aligned}
ik \sin(\alpha) \left(g_3(\alpha) - g_4(\alpha + \pi/4) - g_1(\alpha + \pi/4) + g_2(\alpha) \right) &\equiv -\Gamma \left(g_3(\alpha) + g_4(\alpha + \pi/4) \right) \\
g_3(\alpha) + g_4(\alpha + \pi/4) &\equiv g_1(\alpha + \pi/4) + g_2(\alpha) \\
ik \sin(\alpha) \left(g_1(\alpha) - g_2(\alpha + \pi/4) \right) &\equiv -\gamma \left(g_1(\alpha) + g_2(\alpha + \pi/4) \right) \\
g_{3\pm}(\alpha + \pi/4) \mp g_{2\pm}(\alpha) &\equiv 0
\end{aligned} \tag{9}$$

Here we add the $+$ or $-$ signs to the functions corresponding to v_+ and v_- respectively.

If we introduce the notations $\mathbf{g}(\alpha) \equiv (g_1(\alpha), g_4(\alpha), g_3(\alpha), g_2(\alpha))^t$ the functional equations (9) can be rewritten as

$$\mathbf{g}(\alpha + \pi/4) = M(\alpha)\mathbf{g}(\alpha) \tag{10}$$

where the meromorphic matrix function M is given by

$$M_\pm(\alpha) = \begin{pmatrix}
0 & 0 & \frac{2ik \sin(\alpha)}{2ik \sin(\alpha) - \Gamma} & \frac{\Gamma}{2ik \sin(\alpha) - \Gamma} \\
0 & 0 & \frac{\Gamma}{2ik \sin(\alpha) - \Gamma} & \frac{2ik \sin(\alpha)}{2ik \sin(\alpha) - \Gamma} \\
0 & \pm 1 & 0 & 0 \\
\frac{ik \sin(\alpha) + \gamma}{ik \sin(\alpha) - \gamma} & 0 & 0 & 0
\end{pmatrix}. \tag{11}$$

To study the functional equations (10) we need some algebraic results obtained in our previous papers [1, 13] for the generalized class of similar functional equations. Below we apply these results to the problem under consideration. Namely, let $M(x)$

and $F(x)$ be meromorphic matrices depending on the complex variable x and such that for some $n \in \mathbf{N}$

$$M(x+n) = M(x); \quad F(x+n) = F(x). \tag{12}$$

The functional equation (10) with the argument shift $\pi/4$ can be reduced to the equation with the argument shift 1 and the matrix coefficient having the period n instead of 2π. The properties of matrix coefficient $M(\alpha)$ from (11) can be reconstructed if one poses $n = 8$, $x = 4\alpha/\pi$. We have considered the matrix finite-difference equation

$$g(x+1) = M(x)g(x) + F(x). \tag{13}$$

We introduced the notation for ordered product of some matrices $\{M_i\}$, namely $\uparrow \prod_{i=l}^{k} M_i \equiv M_k M_{k-1}...M_l$ for $k \geq l$ and $\uparrow \prod_{i=l}^{k} M_i = I$ for $k < l$. In this terms the n-th iteration of the equation (13) reads as

$$g(x+n) = N(x)g(x) + \Xi(x) \tag{14}$$

where we denoted $N(x) \equiv \uparrow \prod_{i=0}^{n-1} M(x+i)$ and $\Xi(x) \equiv \sum_{i=0}^{n-1} \uparrow \prod_{j=i+1}^{n-1} M(x+j) F(x+i)$. Due to the definition of N one has

$$N(x+n) = N(x); \quad \Xi(x+n) = \Xi(x) \tag{15}$$

$$N(x+1) = M(x)N(x)M^{-1}(x). \tag{16}$$

So the eigenvalues of $N(x)$ are of the period 1. Then we considered two special cases: a) $N(x) \equiv I$ b) $\det(N(x) - I) \not\equiv 0$ which covered the three body problem with zero-range interactions and are suitable for the problem we consider. We need the following statements [1, 13]:

Lemma 1 *The general solution of inhomogeneous functional equation (14) corresponding to the cases a) and b) looks like*

$$\begin{array}{ll} a) & g(x) = \frac{x}{n}\Xi(x) + g_0(x); \\ b) & g(x) = (I - N(x))^{-1}\Xi(x) + g_0(x), \end{array} \tag{17}$$

where $g_0(x)$ is a solution of homogeneous functional equation

$$g_0(x+n) = N(x)g_0(x) \tag{18}$$

Using this statement we were able to proof the

Theorem 1 *The general solution of inhomogeneous functional equation (13) for the cases a) and b) looks like*

$$a) \quad \begin{array}{l} g(x) = \sum_{i=0}^{n-1} \uparrow \prod_{j=i+1}^{n-1} M(x+j) \left(\frac{x+i}{n}\right) F(x+i) + \\ + \sum_{i=0}^{n-1} \uparrow \prod_{j=i}^{n-1} M(x+j) g_0(x+i), \end{array} \tag{19}$$

$$b) \quad \begin{array}{l} g(x) = (I - N(x))^{-1}\Xi(x) + \\ + \sum_{i=0}^{n-1} \uparrow \prod_{j=i}^{n-1} M(x+j) g_0(x+i), \end{array} \tag{20}$$

where $g_0(x)$ is a solution of homogeneous functional equation (18). For the case $\det N(x) \not\equiv 0$ the solution can be also expressed as

$$g(x) = \sum_{i=0}^{n-1} \uparrow \prod_{j=i}^{n-1} M(x+j)(I - N(x+i))^{-1} F(x+i) + $$
$$+ \sum_{i=0}^{n-1} \uparrow \prod_{j=i}^{n-1} M(x+j) g_0(x+i). \tag{21}$$

The statements above show that the first step in the study of the equations (9) should be the construction of the 8-th iteration of the matrix $M(\alpha)$. In order to simplify the calculations we introduce the notations

$$\Gamma_k := \Gamma/k$$
$$\gamma_k := 2\gamma/k \tag{22}$$
$$z := e^{i\alpha}$$

and represent M as $M = \begin{pmatrix} 0 & Q/q \\ P/p & 0 \end{pmatrix}$ where

$$P_{\pm}(z) = \begin{pmatrix} 0 & \pm(z^2 - \gamma_k z - 1) \\ z^2 + \gamma_k z - 1 & 0 \end{pmatrix}$$
$$p(z) = z^2 - \gamma_k z - 1$$
$$Q(z) = \begin{pmatrix} z^2 - 1 & \Gamma_k z) \\ \Gamma_k z & z^2 - 1 \end{pmatrix} \tag{23}$$
$$q(z) = z^2 - \Gamma_k z - 1.$$

This allows us, after one iteration, to deal with the 2×2 matrices instead of 4×4 ones, and consider the equation $\mathbf{g}(\alpha + 2\pi) = N(\alpha)\mathbf{g}(\alpha)$ as a set of two independent equations.

4   Anti-symmetric case and the Bethe ansatz

Iteration of the equations for v_-, i.e. the case which is center-mass anti-symmetric with respect to the barrier, shows a surprising result:

$$N_-(\alpha) \equiv I.$$

As it was mentioned in [1] the latter result means that the Bethe ansatz can be used for the exact solution of the problem. One can construct this solution through either SM transformation (see e.g. [9]) or directly solving the finite linear system for the plane waves coefficients. Here we give the result of these calculations.

We shall construct the solution as the decomposition of plane waves. Let us fix the wave vector (k_1, k_2) for one of the plane waves. Using the reflection rules we conclude that the complete set of the plane waves can be written as:

$$\phi \in (0, \pi/4) \quad v_-(x, y) = \sum_{l=1,...,8} A_l e^{i\mathbf{k}_l \mathbf{x}}$$
$$\phi \in (\pi/4, \pi/2) \quad v_-(x, y) = \sum_{l=1,...,8} B_l e^{i\mathbf{k}_l \mathbf{x}} \tag{24}$$

where

$$\{\mathbf{k}_l\}_{l=1,..,8} = \left\{ \begin{array}{rr} (& k_1, \quad k_2) \\ (& -k_1, \quad -k_2) \\ (& k_1, \quad -k_2) \\ (& -k_1, \quad k_2) \\ (& k_2, \quad k_1) \\ (& -k_2, \quad -k_1) \\ (& k_2, \quad -k_1) \\ (& -k_2, \quad k_1) \end{array} \right\} \tag{25}$$

and $\mathbf{x} := (x,y)$.

Let $\Re k_1 \leq 0$, $\Re k_1 \leq 0$ and $\Re k_1 < \Re k_2$, that is the wave vectors (k_1, k_2) and (k_2, k_2) correspond to the incoming waves which represent two situations: first, the particles are coming from the same side of the barrier; second the particles are coming from different sides of the barrier. To get the scattering amplitudes one should solve the boundary value problem (7) for $A_1 = 1; B_5 = 0$ (first case) and $A_1 = 0; B_5 = 1$ (second case). The results we obtained for the scattering amplitudes are given by equations (26) and (27).

At $A_1 = 1;\ B_5 = 0$ we have:

$$
\begin{aligned}
A_2 &= \frac{2\gamma k_1{}^2(\gamma+ik_2)+2k_1{}^3(-i\gamma+k_2)+k_1(-i\gamma+k_2)\left(\Gamma^2-2k_2{}^2\right)-\gamma(\gamma+ik_2)\left(\Gamma^2+2k_2{}^2\right)}{(\gamma-ik_1)(\gamma-ik_2)\left(i\Gamma+\sqrt{2}k_1-\sqrt{2}k_2\right)\left(i\Gamma+\sqrt{2}k_1+\sqrt{2}k_2\right)} \\
A_3 &= -\frac{\gamma+ik_2}{\gamma-ik_2} \\
A_4 &= \frac{-2\gamma k_1{}^2+2ik_1{}^3+ik_1\left(\Gamma^2-2k_2{}^2\right)+\gamma\left(\Gamma^2+2k_2{}^2\right)}{(\gamma-ik_1)\left(i\Gamma+\sqrt{2}k_1-\sqrt{2}k_2\right)\left(i\Gamma+\sqrt{2}k_1+\sqrt{2}k_2\right)} \\
A_5 &= -\frac{\Gamma}{\Gamma-i\sqrt{2}(k_1-k_2)} \\
A_6 &= -\frac{\Gamma(\gamma+ik_1)(\gamma+ik_2)}{(\gamma-ik_1)(\gamma-ik_2)\left(\Gamma-i\sqrt{2}k_1-i\sqrt{2}k_2\right)} \\
A_7 &= \frac{\Gamma(\gamma+ik_1)}{(\gamma-ik_1)\left(\Gamma-i\sqrt{2}k_1+i\sqrt{2}k_2\right)} \\
A_8 &= \frac{\Gamma(\gamma+ik_2)}{(\gamma-ik_2)\left(\Gamma-i\sqrt{2}(k_1+k_2)\right)} \\
B_1 &= \frac{i\sqrt{2}(k_1-k_2)}{\Gamma-i\sqrt{2}(k_1-k_2)} \\
B_2 &= \frac{\sqrt{2}(\gamma+ik_2)(k_1+k_2)}{(\gamma-ik_2)\left(i\Gamma+\sqrt{2}k_1+\sqrt{2}k_2\right)} \\
B_3 &= -\frac{\sqrt{2}(\gamma+ik_2)(k_1+k_2)}{(\gamma-ik_2)\left(i\Gamma+\sqrt{2}k_1+\sqrt{2}k_2\right)} \\
B_4 &= \frac{i\sqrt{2}(k_1-k_2)}{\Gamma-i\sqrt{2}(k_1-k_2)} \\
B_6 &= \frac{2\sqrt{2}\Gamma k_1(-i\gamma+k_2)}{(\gamma-ik_1)\left(i\Gamma+\sqrt{2}k_1-\sqrt{2}k_2\right)\left(i\Gamma+\sqrt{2}k_1+\sqrt{2}k_2\right)} \\
B_7 &= \frac{2i\sqrt{2}\Gamma k_1(\gamma+ik_2)}{(\gamma-ik_1)\left(i\Gamma+\sqrt{2}k_1-\sqrt{2}k_2\right)\left(i\Gamma+\sqrt{2}k_1+\sqrt{2}k_2\right)} \\
B_8 &= 0
\end{aligned}
\tag{26}
$$

At $A_1 = 0$; $B_5 = 1$ we have:

$$B_1 = -\frac{\Gamma}{\Gamma - i\sqrt{2}\,k_1 + i\sqrt{2}\,k_2}$$

$$B_2 = \frac{-i\,\Gamma}{i\,\Gamma + \sqrt{2}\,k_1 + \sqrt{2}\,k_2}$$

$$B_3 = \frac{i\,\Gamma}{i\,\Gamma + \sqrt{2}\,k_1 + \sqrt{2}\,k_2}$$

$$B_4 = \frac{\Gamma}{\Gamma - i\sqrt{2}\,k_1 + i\sqrt{2}\,k_2}$$

$$B_6 = \frac{2\,\gamma\,k_1{}^2 + 2\,i\,k_1{}^3 + i\,k_1\left(\Gamma^2 - 2\,k_2{}^2\right) - \gamma\left(\Gamma^2 + 2\,k_2{}^2\right)}{(\gamma - i\,k_1)\left(i\,\Gamma + \sqrt{2}\,k_1 - \sqrt{2}\,k_2\right)\left(i\,\Gamma + \sqrt{2}\,k_1 + \sqrt{2}\,k_2\right)}$$

$$B_7 = \frac{-2\,\gamma\,k_1{}^2 - 2\,i\,k_1{}^3 - i\,k_1\left(\Gamma^2 - 2\,k_2{}^2\right) + \gamma\left(\Gamma^2 + 2\,k_2{}^2\right)}{(\gamma - i\,k_1)\left(i\,\Gamma + \sqrt{2}\,k_1 - \sqrt{2}\,k_2\right)\left(i\,\Gamma + \sqrt{2}\,k_1 + \sqrt{2}\,k_2\right)}$$

$$B_8 = -1$$

$$A_2 = \frac{2\sqrt{2}\,\Gamma\,k_1\left(-i\,\gamma + k_2\right)}{(\gamma - i\,k_1)\left(i\,\Gamma + \sqrt{2}\,k_1 - \sqrt{2}\,k_2\right)\left(i\,\Gamma + \sqrt{2}\,k_1 + \sqrt{2}\,k_2\right)}$$

$$A_3 = 0$$

$$A_4 = \frac{2\sqrt{2}\,\Gamma\,k_1\left(i\,\gamma + k_2\right)}{(\gamma - i\,k_1)\left(i\,\Gamma + \sqrt{2}\,k_1 - \sqrt{2}\,k_2\right)\left(i\,\Gamma + \sqrt{2}\,k_1 + \sqrt{2}\,k_2\right)}$$

$$A_5 = \frac{\sqrt{2}\,(k_1 - k_2)}{i\,\Gamma + \sqrt{2}\,k_1 - \sqrt{2}\,k_2}$$

$$A_6 = \frac{\sqrt{2}\,(\gamma + i\,k_1)(k_1 + k_2)}{(\gamma - i\,k_1)\left(i\,\Gamma + \sqrt{2}\,k_1 + \sqrt{2}\,k_2\right)}$$

$$A_7 = -\frac{\sqrt{2}\,(\gamma + i\,k_1)(k_1 - k_2)}{(\gamma - i\,k_1)\left(i\,\Gamma + \sqrt{2}\,k_1 - \sqrt{2}\,k_2\right)}$$

$$A_8 = -\frac{\sqrt{2}\,(k_1 + k_2)}{i\,\Gamma + \sqrt{2}\,k_1 + \sqrt{2}\,k_2}$$

$$\text{(27)}$$

We are not able now to discuss the physical meaning of the results obtained in (26) and (27) because we need the similar expressions for the v_+ as part of the scattering amplitude. This is what we plan to do separately. However, some basic features of solutions obtained with (26) and (27) can be already discussed. Here we shall pay attention to the scattering of the particles being initially bound in a cluster. The incoming cluster is represented by the incoming surface wave:

$$e^{-ikx - \gamma y}; \lambda = k^2 - \gamma^2.$$

Substituting this into (26) one has at $A_1 = 1$ and $B_5 = 0$

$$B_1 = 1 + \frac{\Gamma}{i\sqrt{2}\,k + \sqrt{2}\,\gamma - \Gamma}$$

$$B_2 = 0 \quad B_3 = 0$$

$$B_4 = -1 + \frac{\Gamma}{-i\sqrt{2}\,(k - i\,\gamma) + \Gamma}$$

$$B_6 = 0 \quad B_7 = 0 \quad B_8 = 0$$

$$A_2 = 0 \quad A_3 = 0 \quad A_4 = -\frac{(k - i\,\gamma)\left(2\,k^2 + 4\,i\,k\,\gamma - 2\,\gamma^2 + \Gamma^2\right)}{(k + i\,\gamma)\left(\sqrt{2}\,k - i\sqrt{2}\,\gamma + i\,\Gamma\right)\left(\sqrt{2}\,k + i\left(\sqrt{2}\,\gamma + \Gamma\right)\right)}$$

$$A_5 = \frac{\Gamma}{i\sqrt{2}\,k + \sqrt{2}\,\gamma - \Gamma}$$

$$A_6 = 0$$

$$A_7 = \frac{(i\,k + \gamma)\,\Gamma}{(-i\,k + \gamma)\left(-i\sqrt{2}\,k - \sqrt{2}\,\gamma + \Gamma\right)}$$

$$A_8 = 0$$

$$\text{(28)}$$

Considering (26) we should first mention that the system in question does not allow rearrangements, i.e. as the coupled particles come to the barrier they scatter

back also coupled. No transitions like: $2 \leftrightarrow 1+1$; $2 \leftrightarrow 1+1^*$ is possible. Here * stands for the particle bound at the barrier. This exceptional, in general, but common for the Bethe ansatz situation is due to the symmetry restrictions we applied. The coefficient A_4 which plays the role of the reflection coefficient depends effectively on the strength of both interaction with the barrier and the interaction between particles. However in order to study the influence of both interactions on the cluster conductance through the barrier one should combine the data obtained for v_- with ones for v_+. In the next section we shall start the solution of the scattering problem for v_+.

5 Symmetric case and the Riemann-Hilbert Problem

The symmetric case, i.e. the case which is center-mass symmetric with respect to the barrier, leads to the following expression

$$N(z) = \begin{pmatrix} N_1(z)/p_1(z) & 0 \\ 0 & N_2(z)/p_2(z) \end{pmatrix} \tag{29}$$

through the 2×2 matrixes $N_1(z)$ and $N_2(z)$ with polynomial coefficients and polynomials $p_1(z)$ and $p_2(z)$. We do not present here the exact expressions for $N_i(z)$ and $p_i(z)$ we obtained by using the computerized tool for analytical calculations (*Mathematica*). Instead of this we shall adopt our general results [1, 13] to the case of the functional equation with 2×2 rational matrix coefficient as it has been done in (29). In this way we can not obtain the exact solution for v_+ as it was done for v_-, however we are able to study the properties of this solution in much details. Namely, for the application to the solution of scattering problem one needs to find analytic solutions of functional equations (13) with given singularities, which would correspond to the incoming plane waves [10, 2]. In order to do this it is useful to find a generating solution, i.e. a meromorphic matrix function having meromorphic inverse one with finite number of singularities in the band $\Re x \in [0, 1)$. Then the solution with given properties can be obtained through the action of generating solution on some periodic vector function which is simply a rational function of exponentials. A generating solution for the functional equations (13) can be obtained from the solution of the iterated equations (14) using the results of Theorem 1. So in the rest of the paper we shall concentrate on the functional equations (14) only.

We reformulate the equations (14) to the form of the boundary value problem by introducing the variable $z = e^{2\pi i \frac{x}{n}}$, which coincide with the definition (22) in the case period 2π.

Due to (15) the matrix function $N(z)$ will be meromorphic, which is just obvious in our case, and the boundary values of solutions for (14) on the cut $z \in [0, +\infty)$ will satisfy

$$g(z - i0) = N(z) g(z + i0) + \Xi(z). \tag{30}$$

This is the special case of the problem known as the matrix Riemann-Hilbert problem

. We propose the way to solve this type of boundary value problems and demonstrate the functional class of solutions.

Let us define $\mathcal{L} := [0, +\infty)$, $z_{l-} := 0$, $z_{l+} := +\infty$, $\ln_{\mathcal{L}} z := \ln z$. Let us also note that the requirement $N(z_{l\pm}) = I$ is obviously fulfilled in our case. Here we need the results obtained for the **Homogeneous Riemann-Hilbert Problem:** *Find the vector function* $\mathbf{g}(z) | \mathbf{C} \mapsto \mathbf{C}^n$ *analytic on* $\overline{\mathbf{C}} \backslash \{\mathcal{L}\}$, *extendable until the both sides of* $\mathcal{L}$ *and such that its boundary values satisfy*

$$\mathbf{g}(z)|_{\mathcal{L}_+} = N(z)\mathbf{g}(z)|_{\mathcal{L}_-}. \tag{31}$$

The solution of an inhomogeneous problem would be needed if we study some singular solutions for v_+ (compare with [9]).

In our consideration we use the matrix coefficient $N(z)$ in the diagonal (or Jordan) form. Here we can omit the $n \times n$ case, which we briefly considered in our previous paper [1] and come directly to the simple diagonalization of a 2×2 matrix N. Below we shall use the symbol $N(z)$ to refer any of N_1/p_1 or N_2/p_2. Since any 2×2 matrix has just two eigenvalues, which are given by the values of the two-sheet analytic function

$$\lambda(z) = \frac{1}{2}\left(\mathrm{Tr}N(z) + \sqrt{\mathrm{Tr}^2 N(z) - 4\mathrm{Det}N(z)}\right),$$

the procedure of finding the diagonal or Jordan form is direct. Moreover, for the problem in question we can find explicitly the expressions for

$$\begin{aligned}
&\mathrm{Tr}^2 N_1(z) - 4\mathrm{Det}N_1(z) = \\
&-128z^4\left(-1 + z^4\right)^2 \gamma^2 \Gamma^2 \\
&(1 + z^{16} + (z^4 + z^{12})(-\gamma^4 + \Gamma^2(4 + \Gamma^2) - 4\gamma^2(1 + 2\Gamma^2)) - \\
&z^8(4\gamma^2(2 + \Gamma^4) + 2(1 + 4\Gamma^2 + \Gamma^4) + \gamma^4(2 + 4\Gamma^2 + \Gamma^4)))
\end{aligned} \tag{32}$$

and

$$\begin{aligned}
&\mathrm{Tr}^2 N_2(z) - 4\mathrm{Det}N_2(z) = \\
&128z^4\left(1 + z^4\right)^2 \gamma^2 \Gamma^2 \\
&(1 + z^{16} + (z^4 + z^{12})(\gamma^4 - \Gamma^2(4 + \Gamma^2) + 4\gamma^2(1 + 2\Gamma^2)) + \\
&z^8(4\gamma^2(2 + \Gamma^4) + 2(1 + 4\Gamma^2 + \Gamma^4) + \gamma^4(2 + 4\Gamma^2 + \Gamma^4))).
\end{aligned} \tag{33}$$

We conclude that we deal now with the case (1.b) of [1], Let us mention that the eigenvalues $\lambda_1(z) \neq \lambda_2(z)$ a.e. and have branching points.

The corresponding eigenprojections $\Lambda_i(z)$ can be expressed as $\Lambda_i(z) = \mathbf{v}_i(z) \times \widetilde{\mathbf{v}}_i^t(z)$ where $\mathbf{v}_i(z)$ is the eigenvector of $N(z)$ corresponding to the eigenvalue $\lambda_i(z)$ and $\widetilde{\mathbf{v}}_i^t(z)$ is one for $N(z)^t$. These eigenvectors are normalized as $\widetilde{\mathbf{v}}_i^t(z) \times \mathbf{v}_i(z) = 1$, so $\mathrm{Tr}\Lambda_i(z) = 1$, $\Lambda_i^2(z) = \Lambda_i(z)$ and $\Lambda_j(z)\Lambda_i(z) = 0$ for $i \neq j$.

Let us recall the notion of the generating solution we used [1]. We call an analytic matrix function $X(z)$ *generating solution* of the Riemann-Hilbert problem [1] (31) if

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[1]Note that the generating solution for the functional equation (14) can be obtained as $X\left(e^{2\pi i \frac{x}{n}}\right)$.

1. it satisfies the boundary value problem

$$X(z)|_{\mathcal{L}_+} = N(z) X(z)|_{\mathcal{L}_-}. \tag{34}$$

2. $X(z)$ and $X^{-1}(z)$ are regular on $\mathbf{C}/\{\mathcal{L}\}$ except may be some finite number s of points $\{z_i\}_{i=1}^s$ where they poles of finite multiplicities.

Suppose we know some generating solution of the problem (31). One can show that for any regular solution $\mathbf{g}(z)$ the function $\mathbf{g}_0(z) \equiv X^{-1}(z)\mathbf{g}(z)$ should satisfy

$$\mathbf{g}_0(z)|_{\mathcal{L}_+} = \mathbf{g}_0(z)|_{\mathcal{L}_-}$$

and so would be rational function with poles at the points $\{z_i\}$ only and of multiplicity not higher then one of $X^{-1}(z)$. In general, any such function can be expanded in finite linear combination of simplest fractions and the regularity conditions at the points $\{z_i\}$ would lead to the homogeneous system of linear equations for the coefficients in this decomposition. However, we shall demonstration this fact elsewhere.

In [1] we proposed the way how to construct a generating solution for the general case of meromorphic matrix $N(z)$. Here we shall show how this can be applied to the specific problem in question. To find the generating solution we make use of the fact that the matrix coefficient can be represented as

$$N(z) = \lambda_+(z)\Lambda_+(z) + \lambda_-(z)\Lambda_-(z)$$

where the signs $\pm$ denote the values of functions on different sheets of the Riemann surface. In the considered case the structure of the Riemann surface is determined in (32) or (33), i.e. by the algebraic curves

$$K_1(z) = (1 + z^{16} + (z^4 + z^{12})(-\gamma^4 + \Gamma^2(4 + \Gamma^2) - 4\gamma^2(1 + 2\Gamma^2)) - z^8(4\gamma^2(2 + \Gamma^4) + 2(1 + 4\Gamma^2 + \Gamma^4) + \gamma^4(2 + 4\Gamma^2 + \Gamma^4)))^{\frac{1}{2}} \tag{35}$$

or

$$K_2(z) = (1 + z^{16} + (z^4 + z^{12})(\gamma^4 - \Gamma^2(4 + \Gamma^2) + 4\gamma^2(1 + 2\Gamma^2)) + z^8(4\gamma^2(2 + \Gamma^4) + 2(1 + 4\Gamma^2 + \Gamma^4) + \gamma^4(2 + 4\Gamma^2 + \Gamma^4)))^{\frac{1}{2}} \tag{36}$$

which have by 16 branching points $\{h_i\}_{i=1}^a$ in all non-degenerated cases, i.e. $\Gamma \neq 0$, $\gamma \neq 0$. The points h_i are given by simple but bulk expressions we shall leave outside this text. Let us split the set $\{h_i\}_{i=1}^{16}$ in two parts $\{h_i\}_{i=1}^a = \{u_i\}_{i=1}^8 \cup \{v_i\}_{i=1}^8$ and fix the cuts δ_i which run between the points u_i and v_i so that they do not cross the half-axis $[0, \infty)$. Because the construction below is the same for both functions $K_1(z)$ and $K_2(z)$ we shall omit the indices and refer these functions as $K(z)$. We denote by $K_\pm(z)$ two branches of the function $K(z)$. The branches can be fixed by the condition

$$\lim_{z \to \infty} K_+(z)/z^8 = 1.$$

The corresponding branches of $\lambda(z)$ and $\Lambda(z)$ are denoted as $\lambda_\pm(z)$ and $\Lambda_\pm(z)$. If we denote the points of the Riemann surface $\mathcal{R}$ of $K(z)$ by $\mathbf{z} \equiv (z, j)$; $z \in \mathbf{C}$; $j = \pm$ these branches are defined as $\lambda_\pm(z) \equiv \lambda((z, \pm))$; $\Lambda_\pm(z) \equiv \Lambda((z, \pm))$.

Let us define the function $g(\mathbf{z})$ on $\mathcal{R}$

$$g_0(\mathbf{z}) = \exp\left\{ \frac{K(\mathbf{z})}{2\pi i} \int_0^{+\infty} \frac{\ln \lambda_+(z')}{K_+(z')(z'-z)} dz' \right\} \tag{37}$$

We have to mention that (37) gives a new representation of $g_0(\mathbf{z})$ in comparing with $g_0(\mathbf{z})$ from [1]. This form is more compact and seems to be more suitable for the actual calculations since it does not require the calculation of $\lambda(z)$ singularities.

The function $g_0(\mathbf{z})$ is meromorphic on $\mathcal{R}\backslash\{\infty\}$ and has the essential singularity at $z \to \infty$. As $z \to \infty$ the function $g_0(\mathbf{z})$ behaves as $e^{q(z)}$ where the coefficients of the polynomial $q(z)$ of the degree 8.

This singularity can be canceled if one multiplies $g_0(\mathbf{z})$ by a product of the Baker-Ahiezer (BA) function $\psi(\mathbf{z})$ of the Riemann surface $\mathcal{R}$ corresponding to the point ∞, the polynomial $q(z)$ and some non-special divisor D of degree n [19]. Really, if one fixes n points on $\mathcal{R}$ and allows the function $\psi(\mathbf{z})$ to have simple poles at this points, i.e. specifies the divisor D, then the condition $\psi(\mathbf{z}) \sim e^{-q(z)}$ at $\mathbf{z} \to \infty$ will specify the function ψ uniquely up to the constant factor. In order to get the explicit expression for the BA function one chooses the canonical sections $\{a_i\}_{i=1}^8$ and $\{b_i\}_{i=1}^8$ on the Riemann surface $\mathcal{R}$. Then one constructs the second kind differential Ω with the main part $-dq(z)$ at ∞ and normalized by the conditions

$$\oint_{a_i} \Omega = 0.$$

When one denotes by $U = \{U_1, .., U_m\}$ the vector of b-periods of Ω,

$$U_i = \oint_{b_i} \Omega$$

and chooses some point $\mathbf{z}_0$ on $\mathcal{R}$ the explicit expression for ψ looks like [19]

$$\psi(\mathbf{z}) = \exp\left(\int_{\mathbf{z}_0}^{\mathbf{z}} \Omega \right) \frac{\theta(A(\mathbf{z}) - A(D) + U - K)}{\theta(A(\mathbf{z}) - A(D) - K)} \tag{38}$$

where K is the vector of Riemann constants, A denotes the Abel transform and θ is the Riemann theta-function of the surface $\mathcal{R}$.

Let us introduce the function $g(\mathbf{z}) = g_0(\mathbf{z})\psi(\mathbf{z})$. The branches of $g(\mathbf{z})$ on the sheets $\mathcal{R}$ we shall denote by

$$g_+(z) \equiv g((z,+)); \quad g_-(z) \equiv g((z,-)).$$

In this notations the generating solution of the Riemann-Hilbert problem can be found as

$$X(z) = g_+(z)\Lambda_+(z) + g_-(z)\Lambda_-(z).$$

The regularization of $g_0(\mathbf{z})$ described above has some shortcomings. Namely, the investigation of $X^{-1}(z)$ meets the problem of theta-function zeroes and the formulae obtained are not a good base for the numerical calculations. In the concluding section of [1] we proposed another method to construct the BA function. The method leads to the set of Dubrovin's type [19] ordinary differential equations. We plan, in the further study on the subject, to investigate the solutions of this equations for the particular case we study here. However we should emphasize that the method of the solutions construction for (10) and their functional class is already studied.

6 Conclusion

In summary, we have studied a new mathematical model for correlated tunneling of electrons through a barrier in 1D quantum wires. For one subproblem the exact solution has been obtained and for another one the reduction to the generalized Dubrovin's equation has been constructed. The latter can be solved either numerically or analytically. The future work will be devoted to the exact integration of the Dubrovin's equations obtained and to the calculation in terms of their solutions of the physical characteristics of correlated tunneling in quantum wires.

Finally we note that the approach developed in this paper can be easily applied to the phenomena of type 1) mentioned in Introduction, i.e. to the study of exotic exciton complexes in 1D semiconducting nanostructures.

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